Unitary equivalences for essential extensions of C^*-algebras *

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Abstract

Let A be a unital separable C^*-algebra and $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra. Let $\tau : A \to M(B)/B$ be a weakly unital full essential extensions of A by B. We show that there is a bijection between a quotient group of $K_0(B)$ onto the set of strong unitary equivalence classes of weakly unital full essential extensions σ such that $[\sigma] = [\tau]$ in $KK^1(A, B)$. Consequently, when this group is zero, unitarily equivalent full essential extensions are strongly unitarily equivalent. When B is a non-unital but σ-unital simple C^*-algebra with continuous scale, we also study the problem when two approximately unitarily equivalent essential extensions are strongly approximately unitarily equivalent. A group is used to compute the strongly approximate unitary equivalence classes in the same approximate unitary equivalent class of essential extensions.

1 Introduction

Let $\mathcal{K}$ be the C^*-algebra of all compact operators on the Hilbert space l^2 and let X be a compact metric space. Suppose that $\tau_1, \tau_2 : A \to B(l^2)/\mathcal{K}$ are two essential extensions, where $A = C(X)$. The Brown-Douglas-Fillmore theorem states that τ_1 and τ_2 are unitarily equivalent if and only if $[\tau_1] = [\tau_2]$ in $KK^1(A, \mathcal{K})$. Here unitarily equivalent means that there is a partial isometry $v \in B(l^2)/\mathcal{K}$ such that $v^*v = \tau_2(1)$, $vv^* = \tau_1(1)$ and $v^*\tau_1(a)v = \tau_2(a)$ for all $a \in A$. If both are weakly unital, i.e., $\tau_1(1) = \tau_2(1) = 1_{B(l^2)/\mathcal{K}}$, then there exists a unitary $U \in B(l^2)$ such that $\pi(U)^*\tau_1(a)\pi(U) = \tau_2(a)$ for all $a \in A$, if $[\tau_1] = [\tau_2]$, where $\pi : B(l^2) \to B(l^2)/\mathcal{K}$ is the quotient map. In other words, unitary equivalence is same as strong unitary equivalence in this case. This fact is particularly important in the classification of essentially normal operators. This is no longer true if A is not a commutative C^*-algebra in general. Suppose that $A = \mathcal{O}_n$ for $n \geq 3$, it is known that the above is false. In fact that there are $\tau_1, \tau_2 : \mathcal{O}_n \to B(l^2)/\mathcal{K}$ for which there is a unitary $u \in B(l^2)/\mathcal{K}$ such that $\mathrm{ad}\, u \circ \tau_1 = \tau_2$ but they are not strongly unitarily equivalent.

The usual way to get around it is to stabilize the situation. By identifying $M_2(B(l^2)/\mathcal{K})$ with $B(l^2)/\mathcal{K}$, one obtains $v' \in M_2(B(l^2)/\mathcal{K})$ such that $v + v'$ is a unitary in $U_0(M_2(B(l^2)/\mathcal{K}))$ so that there is a unitary $U \in M_2(B(l^2))$ (which is again identified with $B(l^2)$) with $\pi(U) = v + v'$. Hence we have $\pi(U)^*\tau_1(a)\pi(U) =$

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$\tau_2(a)$ for all $a \in A$. Since $KK^1(A, B)$ can not detect the difference between (weak) unitary equivalence and strong unitary equivalence, one may choose to ignore this difference by stabilizing the situation. However, this does not changed the fact that, for example in the above mentioned case that $A = \mathcal{O}_n$, there is no unitary $U \in B(l^2)/\mathcal{K}$ such that $\pi(U)^*\tau_1(a)\pi(U) = \tau_2(a)$ for all $a \in A$. As one sees in the classification of essential normal operators, it is vital sometimes to have the strong unitary equivalence instead of (weak) unitary equivalence.

Let us review the case that $A = C(X)$ and B is a non-unital but σ-unital C^*-algebra. Assume that $M(B)/B$ is a purely infinite simple C^*-algebra (like the case that $B = \mathcal{K}$). Suppose that $\tau : C(X) \to M(B)/B$ is a weakly unital essential extension. Let $x \in K_1(M(B)/B)$ and pick a unitary $u \in M(B)/B$ such that $[u] = x$. Consider $\tau_1 = \mathrm{ad}\, u \circ \tau$. Let $\xi \in X$ be a point. Consider $D = \{\tau(f) : f(\xi) = 0\}$. By a result of Pedersen (see [16]), $D^\perp = \{a \in M(B)/B : bd = db = 0\} \neq \{0\}$. It is a hereditary C^*-subalgebra of $M(B)/B$. A result of S. Zhang says that $M(B)/B$ has real rank zero. Consequently there is non-zero projection $p \in D^\perp$. It is clear that $p\tau(a) = \tau(a)p$ for all $a \in C(X)$. Since $M(B)/B$ is purely infinite simple, a result of Cuntz provides a unitary $w_1 \in p(M(B)/B)p$ such that $[w_1] = [u^*]$. Now put $v = w_1 \oplus (1 - p)$. Then $[v] = [u^*]$. But $v\tau(a) = \tau(a)v$ for all $a \in A$. So $\mathrm{ad}\, v \circ \tau = \tau$. Put $z = vu$. Then $\mathrm{ad}\, z \circ \tau = \tau_1$. But $z \in U_0(M(B)/B)$ and there is a unitary $U \in M(B)$ so that $\pi(U) = z$. In other words we have just sketched the proof of the fact that, in this case, unitary equivalence is the same as strong unitary equivalence (see Theorem 1.9 of [9]. Unfortunately this argument can not be applied to more general C^*-algebras A.

In this short note, we show that there is a K-theoretical obstruction to prevent, in general, the unitary equivalence from being the same as strong unitary equivalence. We will describe it and show that when this K-theoretical obstruction disappears, then the unitary equivalence is indeed the same as the strong unitary equivalence. We find that, for a fix $z \in KK^1(A, B)$, there is a bijection between a quotient of $K_0(B)$ and strong unitary equivalence classes of full essential extensions which are represented by z. When $A = C(X)$, this quotient of $K_0(B)$ is zero which explains K-theoretically the reason why these two unitary equivalence relations are the same. This is a direct application of the Universal Coefficient Theorem and some of more recent results about classification of full essential extensions.

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2 Preliminaries

Definition 2.1. Let A be a unital C^*-algebra and C be a non-unital but σ-unital C^*-algebra. Let $\tau_1, \tau_2 : A \to M(B)/B$ be two essential extensions. Extensions τ_1 and τ_2 are said to be *unitarily equivalent* if there exists a partial isometry $v \in M(B)/B$ such that $v^*v = \tau_2(1_A)$, $vv^* = \tau_1(1_A)$ and

$$v^*\tau_1(a)v = \tau_2(a) \text{ for all } a \in A.$$

The extension τ_1 is said to be *weakly unital* if $\tau_1(1_A) = 1_{M(B)/B}$. It should be noted if τ_2 is also weakly unital and τ_1 and τ_1 are unitarily equivalent, then, in the above, v can be chosen to be a unitary in $M(B)/B$.

The two essential extensions are said to be *strongly unitarily equivalent* if there exists a unitary $U \in M(B)$ such that

$$\mathrm{ad}\, \pi(U) \circ \tau_1 = \tau_2.$$

It is obvious that if two essential extensions are strongly unitarily equivalent then they are unitarily equivalent.

Definition 2.2. Let A and D be two unital C^*-algebras. A homomorphism $h : A \to D$ is said to be *full* if the (closed) ideal generated by $h(a)$ is D for every nonzero $a \in A$.

Let B be a non-unital but σ-unital C^*-algebra. An essential extension $\tau : A \to M(B)/B$ is *full* if τ is a full homomorphism.

If $M(B)/B$ is simple, then all essential extensions are full. If A is simple, then all weakly unital essential extensions are full.

Definition 2.3. (cf. [15]) *Let B be a non-unital but σ-unital C^*-algebra. We say $M(B)/B$ has property (P) if for any full element $b \in M(B)/B$ there exist $x, y \in M(B)/B$ such that $xby = 1$.*

If $M(B)/B$ is simple, then $M(B)/B$ has the property (P). This is always the case if B is purely infinite (see also Remark 2.6 below). It is proved in [15] that, if $B = C \otimes \mathcal{K}$, where $C \cong C(X)$ for some finite CW complex, then $M(B)/B$ has the property (P). Other C^*-algebras which have property (P) are discussed in [15].

Let A be a separable C^*-algebra. A is said to satisfy the Universal Coefficient Theorem if for any σ-unital C^*-algebra C one has the following short exact sequence

$$0 \to ext_{\mathbb{Z}}(K_*(A), K_{*-1}(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\Gamma} Hom(K_*(A), K_*(B)) \to 0.$$

C^*-algebras in the so-called "bootstrap" class $\mathcal{N}$ of amenable C^*-algebras satisfy the Universal Coefficient Theorem (UCT) (by [19]). When A is amenable, one has $KK^1(A, B) = Ext(A, B)$.

The following classification of full extensions was established in [15]

Theorem 2.4. *Let A be a unital separable amenable C^*-algebra which satisfies the Universal Coefficient Theorem. Let $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra so that $M(B)/B$ has property* (P). *Then two full essential extensions $\tau_1, \tau_2 : A \to M(B)/B$ are unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \text{ in } KK^1(A, B).$$

Moreover, for any $z \in KK^1(A, B)$, there exists a full essential extension $\tau : A \to M(B)/B$ such that $[\tau] = z$.

The basic question that we consider in this short note is the following: Suppose that $[\tau_1] = [\tau_2]$ in $KK^1(A, B)$ so that τ_1 and τ_2 are unitarily equivalent. Are they strongly unitarily equivalent? If the answer is negative in general, when are they strongly unitarily equivalent?

For non-stable case, the following was proved in [14].

Theorem 2.5. *Let A a unital separable amenable C^*-algebra which satisfies the UCT and let B be a non-unital but σ-unital simple C^*-algebra for which $M(B)/B$ is simple. Let $\tau_1, \tau_2 : A \to M(B)/B$ be two essential extensions. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if $[\tau_1] = [\tau_2]$ in $KL(A, M(B)/B)$.*

Remark 2.6. *Let B be a non-unital but σ-unital simple C^*-algebra. It is shown in [13] that $M(B)/B$ is simple if and only if B has continuous scale (see [13]. It was also shown that when $M(B)/B$ is simple it is purely infinite (see [13]).*

Definition 2.7. *Let $\{A_n\}$ be a sequence of C^*-algebras. Denote by $l^\infty(\{A_n\})$ the C^*-product of $\{A_n\}$ and $c_0(\{A_n\})$ the C^*-direct sum of $\{A_n\}$. We will also use $q_\infty(\{A_n\})$ for the quotient $l^\infty(\{A_n\})/c_0(\{A_n\})$.*

3 Strong unitary equivalence

Definition 3.1. Let A be a C^*-algebra and B be a unital C^*-algebra. Let $G \subset U(B)$ be a normal subgroup. Suppose that $\phi_1, \phi_2 : A \to B$ are homomorphisms. We say ϕ_1 and ϕ_2 are *G-strongly unitarily equivalent* if there exists a unitary $u \in G$ such that

$$\mathrm{ad}\, u \circ \phi_1(a) = \phi_2(a) \quad \text{for all} \quad a \in A.$$

In the case that $G = U_0(B)$, we simply say that ϕ_1 and ϕ_2 are *strongly unitarily equivalent.*

We say ϕ_1 and ϕ_2 are G-strongly approximately unitarily equivalent if there exists a sequence of unitaries $u_n \in G$ such that

$$\lim_{n\to\infty} \mathrm{ad}\, u_n \circ \phi_2(a) = \phi_1(a) \quad \text{for all} \quad a \in A.$$

If $G = U_0(B)$, we simply say that ϕ_1 and ϕ_2 are *strongly unitarily equivalent*

Lemma 3.2. *Let A be a separable C^*-algebra and B be a unital C^*-algebra and $h_1, h_2 : A \to B$ be two homomorphisms. Suppose that G is a normal subgroup of $U(B)$.*

(1) *Suppose that $h_2 = \mathrm{ad}\, u \circ h_1$ for some $u \in U(B)$. Then h_1 and h_2 are G-strongly unitarily equivalent if and only if there is a unitary $v \in U(B)$ such that $[u] = [v]$ in $U(B)/(U_0(B) + G)$ and $vh_1(a) = h_1(a)v$ for all $a \in A$.*

(2) *Suppose that there exists a sequence of unitaries $u_n \in B$ such that*

$$h_2(a) = \lim_{n\to\infty} \mathrm{ad}\, u_n \circ h_1(a) \quad \text{for all} \quad a \in A.$$

Then h_2 and h_1 are G-strongly approximately unitarily equivalent if and only if there exists a sequence of unitaries $v_k \in B$ such that $[v_k] = [u_{n(k)}]$ in $U(B)/(U_0(B) + G)$ for a subsequence $\{n(k)\}$ and

$$\lim_{k\to\infty} \|v_k h_2(a) - h_2(a) v_k\| = 0 \quad \text{for all} \quad a \in A.$$

Proof. (1) Let $w \in G$ such that $w^* \mathrm{ad}\, u \circ h_1(a) w = h_1(a)$ for all $a \in A$. Put $v = wu$. Then $[v] = [u] \in U(B)/(U_0(B) + G)$ and $vh_1(a) = h_1(a)v$ for all $a \in A$.

Conversely, if there is $v \in U(B)$ with $[u] = [v]$ in $U(B)/(U_0(B) + G)$ such that $vh_1(a) = h_1(a)v$ for all $a \in A$. Put $w = u^*v$. Then $w \in G$. However, we have

$$w^* u^* h_1(a) u w = v^* h_1(a) v = h_1(a) \quad \text{for all} \quad a \in A.$$

(2) Suppose that there exists a sequence of unitaries $w_n \in G$ such that

$$\lim_{n\to\infty} \mathrm{ad}\, w_n \circ h_2(a) = h_1(a).$$

Then one has two subsequences $\{m(k)\}$ and $\{n(k)\}$ such that

$$\lim_{n\to\infty} \mathrm{ad}\, w_{m(k)} u_{n(k)} \circ h_1(a) = h_1(a) \quad \text{for all} \quad a \in A.$$

Choose $v_k = w_{m(k)} u_{n(k)}$. Then $[v_k] = [u_{m(k)}]$ in $U(B)/(U_0(B) + G)$. Moreover

$$\lim_{n\to\infty} \|v_k h_1(a) - h_1(a) v_k\| = 0 \quad \text{for all} \quad a \in A.$$

Conversely, if there exists $v_k \in U(B)$ such that $[v_k] = [u_{n(k)}]$ in $U(B)/(U_0(B)+G)$ such that

$$\lim_{n\to\infty} \|v_k h_2(a) - h_2(a)v_k\| = 0 \text{ for all } a \in A.$$

Put $w_k = u_{n(k)}v_k^*$. Then $w_k \in G$.

Moreover, one checks that

$$\|\mathrm{ad}\, w_k \circ h_1(a) - h_2(a)\| \le \|\mathrm{ad}\, w_k \circ h_1(a) - \mathrm{ad}\, w_k \circ h_2(a)\| + \|\mathrm{ad}\, w_k \circ h_2(a) - h_2(a)\| \to 0,$$

as $n \to \infty$ for every $a \in A$. □

Definition 3.3. Let G and F be two groups and $u \in G$ be a distinguished element. Define

$$H_u(G,F) = \{x \in F : \phi(u) = x, \phi \in Hom(G,F)\}.$$

Let A and B be two C^*-algebras. Suppose that A is unital. We write $H_{[1_A]}(K_0(A), K_i(B)) = H_1(K_0(A), K_i(B))$. If $K_0(A) = \mathbb{Z}/n\mathbb{Z}$ for some $n > 1$ and $K_i(B) = \mathbb{Z}$, then $H_1(K_0(A), K_i(B)) = \{0\}$. If $K_0(A) = \mathbb{Z}$ and $[1_A] = 1$ in $\mathbb{Z}$, then $H_1(K_0(A), K_i(B)) = K_i(B)$. Suppose that $K_0(A) = \mathbb{Z}/p^2\mathbb{Z}$ with $[1_A] = \bar{1}$ and $K_i(B) = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$, where $(p,q) = 1$, then $H_1(K_0(A), K_i(B)) = \mathbb{Z}/p\mathbb{Z}$. If $K_i(B)$ is divisible, then $H_1(K_0(A), K_i(B)) = K_i(B)$.

Proposition 3.4. *Let A be a separable C^*-algebra and let C be a unital C^*-algebra. Let $G \subset U(C)$ be a normal subgroup and let $h_1, h_2 : A \to C$ be two monomorphisms. Suppose that h_1 is unital and $h_2 = \mathrm{ad}\, u \circ h_1$. If h_2 and h_1 are G-strongly unitarily equivalent, then $[u] \in H_1(K_0(A), K_1(C)) + \bar{G}$, where $\bar{G}$ is the image of G in $K_1(C)$.*

Proof. Suppose that h_1 and h_2 are G-strongly unitarily equivalent. Then by 3.2, there is $v \in U(C)$ such that $vh_1(a) = h_1(a)v$ and $[v] = [u]$ in $U(C)/(U_0(C)+G$. Thus we obtain a homomorphism $\Phi : A \otimes C(S^1) \to C$ defined by $\phi(a \otimes f) = h_1(a)f(v)$ for $a \in A$ and $f \in C(S^1)$. Consequently, there is a homomorphism $\phi : K_1(A \otimes C(S^1)) \to K_1(C)$ such that $[\psi(1 \otimes \imath)] = [v]$, where $\imath : S^1 \to S^1$ is the identity map. Since $[(1 \otimes \imath)] = [1_A] \in K_0(A) \subset K_1(A \otimes C(S^1))$, we obtain a homomorphism $\phi : K_0(A) \to K_1(C)$ such that $\phi([1_A]) = [u]$. This implies $[u] \in H_1(K_0(A), K_1(C)) + \bar{G}$.

□

Suppose that A satisfies the UCT and B is a σ-unital C^*-algebra. In what follows $\Gamma : KK^1(A,B) \to Hom(K_*(A), K_{*-1}(B))$ is the surjective map given by the UCT.

Lemma 3.5. *Let A be a separable unital C^*-algebra in $\mathcal{N}$ and B be a σ-unital C^*-algebra. Let $\tau \in KK^1(A,B)$ and $\zeta \in H_1(K_0(A), K_0(B))$. Then, there is an essential extension $\tau_1 : A \otimes C(S^1) \to Q(B \otimes \mathcal{K})$ such that*

$$[\tau_1|_A] = [\tau] \quad \text{and} \quad \Gamma([\tau_1])([1_A \otimes \imath)] = \zeta \quad (\text{in } K_1(Q(B \otimes \mathcal{K})) = K_0(B)),$$

where $\imath(z) = z$ for $z \in S^1$ and $Q(B \otimes \mathcal{K}) = M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$.

Proof. Let $j : A \to A \otimes C(S^1)$ be defined by $j(a) = a \otimes 1$. If $A \otimes C(S^1)$ is identified with $C(S^1, A)$, then $j(a)(t) = a$ for all $t \in S^1$. By the Kunneth formula for tensor product, we obtain $K_0(A \otimes C(S^1)) = K_0(A) \oplus K_1(A)$ and $K_1(A \otimes C(S^1)) = K_1(A) \oplus K_0(A)$. Moreover, $j_{*0} : K_0(A) \to K_0(A) \oplus K_1(A)$ may be written as $j_{*0}(x) = x \oplus 0$ and $j_{*1} : K_1(A) \to K_1(A) \oplus K_0(A)$ may be written as $j_{*1}(y) = y \oplus 0$.

Let $\Gamma(\tau)_i : K_i(A) \to K_{i+1}(B)$ be the map given by the UCT, $i = 0, 1$. Define $\gamma_0 : K_0(A \otimes C(S^1)) = K_0(A) \oplus K_1(A) \to K_1(B)$ by $\gamma_0((x, y)) = \Gamma(\tau)_0(x)$.

Since $\zeta \in H_1(K_0(A), K_0(B))$, there exists $\phi : K_0(A) \to K_0(B)$ such that $\phi([1_A]) = \zeta$. Define $\gamma_1 : K_1(A \otimes C(S^1)) = K_1(A) \oplus K_0(A) \to K_0(B)$ by $\gamma_1((x, y)) = \Gamma(\tau)_1(x) + \phi(y)$. By the UCT, there is $\tau' : A \otimes C(S^1) \to Q(B \otimes \mathcal{K})$ such that $\Gamma(\tau') = (\gamma_0, \gamma_1)$. Define $\psi = \tau'|_A : A \to Q(B \otimes \mathcal{K})$. It follows that $\Gamma(\psi) = \Gamma(\tau' \circ j) = \Gamma(\tau)$. Let $\tau_0 : A \to Q(B \otimes \mathcal{K})$ be an essential extension so that $[\tau_0] = [\tau] - [\psi]$. Then $\Gamma([\tau_0]) = 0$. By the UCT, $[\tau_0] \in ext_{\mathbb{Z}}(K_i(A), K_i(B))$. Suppose that $[\tau_0]$ is represented by the following two short exact sequences:

$$0 \to K_0(B) \to K_0(E) \to K_0(A) \to 0 \quad \text{and} \quad 0 \to K_1(B) \to K_1(E) \to K_1(A) \to 0.$$

Let $G_0 = K_0(E) \oplus K_1(A)$ and $G_1 = K_1(E) \oplus K_0(A)$. Then (G_0, G_1) gives short exact sequences

$$0 \to K_i(B) \to G_i \to K_0(A) \oplus K_1(A) \to 0$$

It gives an element $\delta \in ext_{\mathbb{Z}}(K_i(A \otimes C(S^1)), K_i(B))$. By the UCT again, there is $\sigma : A \otimes C(S^1) \to Q(B \otimes \mathcal{K})$ such that $[\sigma] = \delta$. Let $\tau_0' = \sigma|_A$. Then τ_0' gives the following short exact sequences

$$0 \to K_0(B) \to K_0(E) \to K_0(A) \to 0 \quad \text{and} \quad 0 \to K_1(B) \to K_1(E) \to K_1(A) \to 0$$

and $[\tau_0'] = [\tau_0]$. There is $\sigma' : A \otimes C(S^1) \to Q(B \otimes \mathcal{K})$ such that $[\sigma'] = -[\sigma]$. There is $\tau_1 : A \otimes C(S^1) \to Q(B \otimes \mathcal{K})$ so that $[\tau_1] = [\tau'] - [\sigma]$. Let $\tau'' : A \otimes C(S^1) \to Q(B \otimes \mathcal{K})$ so that $[\tau''] = -[\tau']$. Then $\Gamma(\tau_1) = \Gamma(\tau')$. Moreover,

$$[\tau] - [\tau_1|_A] = [\tau] + [\tau''|_A] + [\sigma'|_A] = [\tau] - [\psi] + [\sigma'|_A] = [\tau_0] + [\sigma'|_A] = 0.$$

Note that $\Gamma([\tau_1]([1_A \otimes \imath]) = \phi([1_A]) = \zeta$. $\square$

Lemma 3.6. *Let $B = C \otimes \mathcal{K}$, where C is unital C^*-algebra and $M(B)/B$ has property (P). Let $u \in U(M(B)/B)$ be a unitary so that $[u] = 0$ in $K_1(M(B)/B)$. Suppose that $\phi : C(S^1) \to M(B)/B$ defined by $\phi(f) = f(u)$ for $f \in C(S^1)$ is full. Then $u \in U_0(M(B)/B)$.*

Moreover, if $z \in K_1(M(B)/B)$, then there exists $u \in U(M(B)/B)$ such that $[u] = z$.

Proof. Let $v \in U_0(M(B)/B)$ be a unitary with $sp(v) = S^1$. Define a monomorphism $\phi_0 : C(S^1) \to M_2(M(B)/B)$ by $\phi_0(f) = f(v) \oplus f(1) \cdot e$, where $e = 1_{M(B)/B}$. Since B is stable, there is $z \in M_2(M(B)/B)$ such that $zz^* = e \oplus e$, $z^*z = e \oplus 0$. Put $\phi_1 = \mathrm{ad}\, z \circ \phi_0$. Put $v_0 = \phi_1(\imath)$, where $\imath$ is the identity function on the unit circle. Since ϕ is full, by 2.17 of [15], it is absorbing. In particular, there exists $W \in M_2(M(B)/B)$ with $W^*W = e \oplus 0$ and $WW^* = e \oplus e$ such that $\mathrm{ad}\, W \circ \phi_1 = \phi$. Therefore $u = W^*(u \oplus v_0)W$. Thus, there is $W_1 \in M_2(M(B)/B)$ with $W_1^*W_1 = e$, $W_1W_1^* = e \oplus e$ such that

$$u = W_1^*(u \oplus e)W_1.$$

Since B is stable and $[u] = 0$ in $K_1(M(B)/B)$, we may assume that $\mathrm{diag}(u, e) \in U_0(M_2(M(B)/B))$. We may write $\mathrm{diag}(u, e) = \prod_{k=1}^n \exp(ia_k)$, where $a_k \in M_2(M(B)/B)$ are self-adjoint elements. It follows that $u = W_1^*(\prod_{k=1}^n \exp(ia_k))W_1$ is connected to $W_1^*W_1 = e$ by a continuous path of unitaries in $U(M(B)/B)$. So $u \in U_0(M(B)/B)$.

To see the last part of the statement, by [15], there is a full essential extension $\tau : C(S^1) \to M(B)/B$ such that $[\tau(\imath)] = z$, where $\imath : S^1 \to S^1$ is the canonical unitary in $C(S^1)$.

$\square$

In the following theorem, let $z \in KK^1(A,B)$ and define $\mathcal{T}_s^1(z)$ to be the set of strong unitary equivalence classes of weakly unital full extensions represented by z.

Theorem 3.7. *Let A be a unital separable C^*-algebra in $\mathcal{N}$ and let $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra so that $M(B)/B$ has property (P). Let $\tau : A \to M(B)/B$ be a weakly unital full essential extension. Suppose that $u \in U(M(B)/B)$. Then $\mathrm{ad}\, u \circ \tau$ is strongly unitarily equivalent to τ if and only if $u \in H_1(K_0(A), K_0(B))$.*

Moreover, there is a bijection

$$\kappa : K_0(B)/H_1(K_0(A), K_0(B)) \to \mathcal{T}_s^1([\tau]).$$

Proof. By 3.4, one needs to prove the "if" part of the statement.

Assume that $[u] \in H_1(K_0(A), K_0(B))$. Then, by 3.5, there is an essential extension $\sigma : A \otimes C(S^1) \to Q(C \otimes \mathcal{K})$ such that $[\sigma|_A i] = [\tau]$ and $[\sigma(1 \otimes \imath)] = [u]$. It follows from 2.17 of [15] that we may assume that σ is full. Since τ is weakly unital, by replacing σ by $\mathrm{ad}\, w_1 \circ \sigma$ for some isometry, we may assume that σ is also weakly unital. By applying 2.17 of [15], there is $w \in U(M(B)/B)$ such that

$$\mathrm{ad}\, w \circ \sigma|_A = \tau.$$

Put $v = \mathrm{ad}\, w \circ \sigma(1 \otimes \imath)$. Then $[v] = [u]$ and $v\tau(a) = \tau(a)v$. It follows from 3.4 that $\mathrm{ad}\, u \circ \tau$ is strongly unitarily equivalent to τ.

Fix $z \in KK^1(A,B)$ for which $z = [\tau]$. We define a map $\kappa : K_0(B)/H_1(K_0(A), K_0(B)) \to \mathcal{T}_s^1(z)$ as follows. For each $x \in K_0(B)/H_1(K_0(A), K_0(B))$, choose $u \in U(M(B)/B)$ such that the image of $[u]$ in $K_0(B)/H_1(K_0(A), K_0(B))$ is x. Define $\kappa(x)$ to be the strong unitary equivalence class represented by $\mathrm{ad}\, u \circ \tau$. Let $u_1, u_2 \in M(B)/B$. We have shown that $\mathrm{ad}\, u_1 \circ \tau$ and $\mathrm{ad}\, u_2 \circ \tau$ are strongly unitarily equivalent if only if $[u_1^* u_2] \in H_1(K_0(A), K_0(B))$. This implies that κ is well defined and is injective. Since for any $\sigma \in \mathcal{T}_s^1(z)$, there is $u \in M(B)/B$ such that $\sigma = \mathrm{ad}\, u \circ \tau$. This shows that κ is also surjective. □

Corollary 3.8. *Let A be a separable C^*-algebra in $\mathcal{N}$ and $B = C \otimes \mathcal{K}$, where C is unital, such that $M(B)/B$ has property (P). Suppose that $K_0(B) = H_1(K_0(A), K_0(B))$. Suppose that $\tau_1, \tau_2 : A \to M(B)/B$ are two weakly unital full essential extensions. Then they are strongly unitarily equivalent if and only if they are unitarily equivalent.*

Corollary 3.9. *In 3.8, if $K_0(A) = G \oplus \mathbb{Z}$ with $[1_A] = (0,1)$, then two weakly unital full essential extensions are strongly unitarily equivalent if and only if they are unitarily equivalent.*

Proof. There is a homomorphism $h : K_0(A) \to \mathbb{Z}$ which maps $[1_A]$ to 1 in $\mathbb{Z}$. For any $\xi \in K_0(B)$, define $\kappa : \mathbb{Z} \to K_0(B)$ by $\kappa(n) = n\xi$. Put $\phi = \kappa \circ h$. Thus $\xi \in H_1(K_0(A), K_0(B))$. Therefore $K_0(B) = H_1(K_0(A), K_0(B))$. Then 3.8 applies.

□

Corollary 3.10. *In 3.8, if A is a unital separable commutative C^*-algebra, then two weakly unital full essential extensions are strongly unitarily equivalent if and only if they are unitarily equivalent.*

Proof. Since $A = C(X)$, $K_0(A) = G \oplus \mathbb{Z}$ with $[1_A] = (0,1)$. □

Let A be a unital C^*-algebra. If $\tau : A \to M(B)/B$ is not unital, then there are fewer strong unitary equivalence classes of full essential extensions $\sigma : A \to M(B)/B$ with $[\tau] = [\sigma]$ which are not weakly unital.

Definition 3.11. Let $B = C \otimes \mathcal{K}$ and A is unital. Suppose that $\tau : A \to M(B)/B$ is not unital and $p = \tau(1_A)$. Let $G_p = \{z \in U(M(B)/B) : z = p + v, v^*v = vv^* = 1 - p\}$. Then G_p is a subgroup. Denote by $\overline{G_p}$ the image of G_p in $K_0(M(B)/B)$. Note that if p is full, then there exists a projection $e \le 1 - p$ and $w \in M(B)/B$ such that $w^*w = e$ and $ww^* = 1_{M(B)/B}$. Then $\overline{G_p} = K_1(M(B)/B)$.

Let $\sigma : A \to M(B)/B$ be another extension. Suppose that $\tau(1_A) = p$ and $\sigma(1_A) = q$. Note that if p is not unitarily equivalent to q (in $M(B)/B$), then τ and σ can not be possibly strongly unitarily equivalent.

From the following result, one also knows that if p and q are unitarily equivalent and $1 - p$ is full, then σ and τ are strongly unitarily equivalent if $[\sigma] = [\tau]$ in $KK^1(A, B)$.

Theorem 3.12. *Let A be a unital separable C^*-algebra in $\mathcal{N}$ and $B = C \otimes \mathcal{K}$, where C is unital for which $M(B)/B$ has property (P). Let $\tau : A \to M(B)/B$ be a full essential extension.*

If $\tau(1_A) = p$, and $p \neq 1$, then there is a bijection from $K_0(B)/(H_1(K_0(A), K_0(B)) + \overline{G_p})$ onto the set of strong unitary equivalence classes of full essential extensions $\sigma : A \to M(B)/B$ for which $[\sigma] = [\tau]$ and $\sigma(1_A)$ is unitarily equivalent to p.

Proof. Suppose that $\sigma(1_A)$ is unitarily equivalent to p and $[\sigma] = [\tau]$. Then there exists a unitary $w \in M(B)/B$ such that $\sigma_1 = \mathrm{ad}\, w \circ \tau$. Suppose that $[w] \in H_1(K_0(A), K_0(B)) + \overline{G_p}$. There is $z \in U(M(B)/B)$ such that $z = p + v$ with $v^*v = vv^* = 1 - p$ and $[zw] \in H_1(K_0(A), K_0(B))$. Note that $\mathrm{ad} zw \circ \tau = \sigma_1$. Thus we may assume that $[w] \in H_1(K_0(A), K_0(B))$. It follows from 3.7 that σ and τ are strongly unitarily equivalent.

On the other hand, note since p is full, $K_1(p(M(B)/B)p) = K_1(M(B)/B) \cong K_0(B)$. Suppose that $z_1, z_2 \in K_1(p(M(B)/B)p)$ such that $\overline{z_1} \neq \overline{z_2}$ in $K_0(B)/(H_1(K_0(A), K_0(B)) + \overline{G_p})$. It follows from 3.6 that there are unitaries $v_1, v_2 \in p(M(B)/B)p$ such that $[v_1] = z_1$ and $[v_2] = z_2$. Consider extensions $\mathrm{ad}\, v_i \circ \tau : A \to p(M(B)/B)p$ for $i = 1, 2$. It follows from 3.4 that they are not G_p-strongly unitarily equivalent as unital homomorphisms to $p(M(B)/B)p$. It follows that they are not strong unitarily as homomorphisms to $M(B)/B$. □

Proposition 3.13. *Let A be a unital separable C^*-algebra in $\mathcal{N}$ and $B = C \otimes \mathcal{K}$, where C is unital such that $M(B)/B$ has property (P). Suppose that $K_1(M(B)/B) = H_1(K_0(A), K_1(M(B)/B)$. Suppose that $\tau_1, \tau_2 : A \to M(B)/B$ are two weakly unital full essential extensions. Then τ_1 and τ_2 are homotopic if and only if $[\tau_1] = [\tau_2]$ in $KK^1(A, B)$.*

Proof. If τ_1 and τ_2 are homotopic, then $[\tau_1] = [\tau_2]$ in $KK^1(A, B)$.

Conversely if τ_1 and τ_2, by 3.7, they are strongly unitarily equivalent. There exists a unitary $u \in U_0(M(B)/B)$ such that $\mathrm{ad}\, u \circ \tau_1 = \tau_2$. Let $\{u_t : t \in [0, 1]\}$ be a continuous path of unitaries in $M(B)/B$ such that $u_0 = u$ and $u_1 = 1_{M(B)/B}$. Define $\Sigma : A \to C([0, 1], M(B)/B)$ by $\Sigma(a)(t) = \mathrm{ad}\, u_t \circ \tau_1$ for $a \in A$. Then $\Sigma(a)(0) = \tau_2$ and $\Sigma(a)(1) = \tau_1$. □

4 Approximate unitary equivalences

Definition 4.1. *Let $\{x_n\}$ be a sequence of elements in $K_i(B)$. Denote by $H_1^{ap}(K_0(A), K_i(B))$ the subset of those sequences $\{x_n\}$ of $K_i(B)$ such that there exists an increasing sequence of finitely generated subgroups $G_n \subset K_0(A)$ with $[1_A] \in G_n$ and group homomorphisms $h_n : G_n \to K_0(B)$ such that $h_n([1_A]) = x_n$. It forms a subgroup of $\prod K_i(B)$.*

Suppose that $\Pi_i : \prod_{n\in N} K_i(B) \to \prod_{n\in N} K_i(B)/\oplus_{n\in N} K_i(B)$, $i = 0, 1$. *Suppose that* $\xi = \Pi_i(\{x_n\})$ *and* $\xi \in H_1(K_0(A), \prod_{n\in N} K_i(B)/\oplus_{n\in N} K_i(B))$. *Then* $\{x_n\} \in H_1^{ap}(K_0(A), K_i(B))$.

If $K_i(B) = H_1(K_0(A), K_i(B))$, *then* $H_1^{ap}(K_0(A), K_i(B)) = \prod_{n\in N} K_i(B)$.

Recall that, for a unitary u in a unital C^*-algebra A,

$$\mathrm{cel}(u) = \inf\{\sum_{k=1}^{n} \|h_k\| : u = \prod_{k=1}^{n} \exp(ih_k),\, n \in \mathbb{N}, h_k \in A_{s.a}\}$$

and $\mathrm{cel}(A) = \sup_{u\in U(A)} \mathrm{cel}(u)$ (see [18]).

Let $r : \mathbb{N} \to \mathbb{N}$ be a map. Unital C^*-algebra A is said to have K_1-r-cancellation if $u \oplus 1_{M_{r(n)}}$ and $v \oplus 1_{M_{r(n)}}$ are in the same path connected component of $U(M_{n+r(n)})$ for any $u, v \in M_n(A)$ with $[u] = [v]$ in $K_1(A)$.

Proposition 4.2. *Let A be a unital separable amenable C^*-algebra, and C be a unital C^*-algebra and $G \subset U(B)$ be a normal subgroup. Suppose that* $\mathrm{cel}(B) \le L$ *for some $L > \pi$ and B has K_1-r-cancellation (for some $r : \mathbb{N} \to \mathbb{N}$). Let $h, \phi : A \to B$ be two unital monomorphisms. Suppose that there exists a sequence of unitaries $u_n \in U(B)$ such that*

$$\lim_{n\to\infty} \mathrm{u_n} \circ h(a) = \phi(a) \text{ for all } a \in A.$$

Suppose also that h and ϕ are G-strongly approximately unitarily equivalent. Then there exists a subsequence $\{n(k)\}$ such that $[u_{n(k)}] \in H_1^{ap}(K_0(A), K_1(B)) + \prod \bar{G}$, where $\bar{G}$ is the image of G in $K_1(B)$.

If $K_0(A)$ is finitely generated, then one can require that $[u_{n(k)}] \in H_1(K_0(A), K_1(B)) + \bar{G}$.

Proof. There exists a sequence of unitaries $v_k \in B$ such that $[v_k] = [u_{n(k)}]$ in $U(B)/(U_0(B) + G)$ for a subsequence $\{n(k)\}$ and

$$\lim_{k\to\infty} \|v_k h_2(a) - h_2(a)v_k\| = 0 \text{ for all } a \in A.$$

Define a map $\Phi : A \otimes C(S^1) \to l^\infty(B)$ by $\Phi(a \otimes f) = \{h_2(a)f(v_k)\}$ for $f \in C(S^1)$. Let $\Pi : l^\infty(B) \to l^\infty(B)/c_0(B)$ be the quotient map. Then $\Psi = \Pi \circ \Phi : A \otimes C(S^1) \to l^\infty(B)/c_0(B)$ is a homomorphism. Since $\mathrm{cel}(B) \le L$ and B has K_1-r-cancellation, it follows from Proposition 2.1 (3) of [6] that

$$K_1(l^\infty(B)) \subset \prod K_1(B) \quad \text{and} \quad K_1(l^\infty(B)/c_0(B)) \subset \prod K_1(B)/\oplus K_1(B).$$

Let $\xi = \Pi_{*1}([\{v_k\}])$. Then one obtains a homomorphism $\gamma : K_0(A) \to \prod K_1(B)/\oplus K_1(B)$ induced by Ψ which maps $[1_A]$ to ξ. Let $F \subset \gamma(K_0(A))$. It is a countable abelian group. Write $F = \cup_n F_n$, where $F_n \subset F_{n+1}$ and each F_n is finitely generated. It is easy to see that for every finitely generated subgroup F_n, there is a homomorphism $f_n : F_n \to \prod K_1(B)$ such that $\Pi \circ f_n = \mathrm{id}_{F_n}$. Let $p_n : \prod K_1(B) \to K_1(B)$ be the projection on the n coordinate. For each n, there is $m(n)$ such that $p_k \circ f_n \circ \gamma([1_A]) = [v_k]$ if $k \ge m(n)$. We may assume that $m(n+1) > m(n)$. Let $G_n \subset K_0(A)$ be a finitely generated subgroup such that $\gamma(G_n) = F_n$ and $G_n \subset G_{n+1}$. We may also assume that $[1_A] \in G_n$. Define $\phi_n = p_{m(n)} \circ f_n \circ \gamma$. Then $\phi_n : G_n \to K_1(B)$ is a homomorphism and $\phi_n([1_A]) = [v_{m(n)}]$. Thus $\{[v_{m(n)}]\} \in H_1^{ap}(K_0(A), K_1(B))$. It follows that $\{[u_{n(m(k))}]\} \in H_1^{ap}(K_0(A), K_1(B)) + \prod \bar{G}$.

□

Theorem 4.3. *Let A be a unital separable C^*-algebra in $\mathcal{N}$. Let $B = C \otimes \mathcal{K}$, where C is unital such that $M(B)/B$ has property (P). Suppose that $K_0(B) = H_1(K_0(A), K_0(B))$. Then two weakly unital full essential extensions of A by B are strongly approximately unitarily equivalent if and only if they are approximate unitarily equivalent.*

Proof. Since $\mathrm{cel}(M(B)/B) \le \mathrm{cel}(M(B))$, by 5.1.15 of [11], $\mathrm{cel}(M(B)/B) \le 6\pi$. By Lemma 3.6, $M(B)/B$ also has K_1-r-cancellation (with $r : \mathbb{N} \to \mathbb{N}$ is the identity map). Suppose that $\tau_1 : A \to M(B)/B$ is a weakly unital full essential extension. Let $\{u_n\} \subset M(B)/B$ be a sequence of unitaries such that

$$\tau_2(a) = \lim_{n\to\infty} \mathrm{ad}\, u_n \circ \tau_1(a) \ \text{ for all } \ a \in A.$$

For each n, since $[u_n] \in H_1(K_0(A), K_0(B))$, as in the proof of 3.7, one obtains a unitary $w_n \in U(M(B)/B)$ such that $[w_n] = [u_n]$ and $w_n\tau_1(a) = \tau_1(a)w_n$. Now let $v_n = w_n^* u_n$. Note $v_n \in U_0(M(B)/B)$ (by 3.6). Then

$$\tau_2(a) = \lim_{n\to\infty} \mathrm{ad}\, v_n \circ \tau_1(a) \ \text{ for all } \ a \in A.$$

So τ_1 and τ_2 are strongly approximately unitarily equivalent. □

Theorem 4.4. *Let A be a unital separable C^*-algebra in $\mathcal{N}$. Let $B = C \otimes \mathcal{K}$, where C is unital such that $M(B)/B$ has property (P). Suppose that $\tau : A \to M(B)/B$ is a weakly unital full essential extension. Then there is an injection from $K_0(B)/H_1(K_0(A), K_0(B))$ to the set of strongly approximately unitarily equivalent classes of weakly unital full essential extensions $\sigma : A \to M(B)/B$ for which $[\sigma] = [\tau]$ in $KL^1(A, B)$.*

Proof. Let $x_1, x_2 \in K_0(B)/H_1(K_0(A), K_0(B))$. Suppose that $u_i \in U(M(B)/B)$ such that $\overline{[u_i]} = x_i$ in $K_0(B)/H_1(K_0(A), K_0(B))$, $i = 1, 2$. Let $\tau_i = \mathrm{ad}\, u_i \circ \tau$, $i = 1, 2$. Suppose that there is a sequence of unitaries $w_n \in U_0(M(B)/B$ such that $\lim_{n\to\infty} \mathrm{ad}\, w_n \circ \tau_1(a) = \tau_2(a)$ for all $a \in A$. It follows that $\lim_{n\to\infty} \mathrm{ad}\, u_2^* \circ \mathrm{ad}\, w_n \circ \tau_1(a) = \tau(a)$ for all $a \in A$. It follows from 4.2 that there is a subsequence $\{k(n)\}$ such that $\{[u_2^* w_{k(n)} u_1]\} \in H_1^{ap}(K_0(A), K_0(B))$. This implies that $\overline{[u_1]} = \overline{[u_2]}$, or $x_1 = x_2$. □

Lemma 4.5. *Let A be a separable unital C^*-algebra in $\mathcal{N}$ and B be a unital purely infinite simple C^*-algebra. Let $\tau : A \to q_\infty(B) = l^\infty(B)/c_0(B)$ be a unital monomorphism and $\zeta \in H_1(K_0(A), K_1(q_\infty(B)))$. Then, there is full monomorphism $\tau_1 : A \otimes C(S^1) \to q_\infty(B)$ such that*

$$[\tau_1|_A] = [\tau] \ \text{ in } \ KL(A, q_\infty(B)) \quad \text{and} \quad \Gamma([\tau_1])([1_A \otimes \imath)] = \zeta \ \text{in} \quad K_1(q_\infty(B)),$$

where $\imath(z) = z$ for $z \in S^1$.

Proof. It follows from 8.5 and 7.7 of [15] that there is a (group) isomorphism from the approximately unitary equivalence classes of full monomorphisms from A to $q_\infty(B)$ and $KL(A, q_\infty(B))$. Exactly the same proof of 3.5 proves the theorem. One may also use the identification $KK^1(A, Sq_\infty(B))) = KK(A, q_\infty(B))$ and apply 3.5. □

Theorem 4.6. *Let B be a σ-unital simple C^*-algebra with continuous scale and let A be a unital separable amenable C^*-algebra in $\mathcal{N}$.*

(i) *Let $\tau : A \to M(B)/B$ be a weakly unital essential extension and $\{u_n\} \in U(M(B)/B)$ such that $[u_n] \in H_1^{ap}(K_0(A), K_1(M(B)/B))$ and*

$$\sigma(a) = \lim_{n\to\infty} \mathrm{ad}\, u_n \circ \tau(a) \ \text{ for all } \ a \in A,$$

where $\sigma : A \to M(B)/B$ is another essential extension. Then σ and τ are strongly unitarily equivalent.

(ii) *If $\prod_{n\in N} K_1(M(B)/B) = H_1^{ap}(K_0(A), K_1(M(B)/B))$ (in particular, if $K_1(M(B)/B) = H_1(K_0(A), K_1(M(B)/B)))$, then two weakly unital essential extensions $\tau_1, \tau_2 : A \to M(B)/B$ are strongly approximately unitarily equivalent if and only if $[\tau_1] = [\tau_2]$ in $KL(A, M(B)/B)$.*

(iii) *If $\tau : A \to M(B)/B$ is a weakly unital essential extension, then there is an injection from $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B)))$ to the set of strong approximate unitary equivalence classes of weakly unital essential extensions σ for which $[\sigma] = [\tau]$ in $KL(A, M(B)/B)$;*

(iv) *In* (iii), *if furthermore, $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ is finite, then there is a bijection from $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ onto the set of strong approximate unitary equivalence classes of weakly unital essential extensions σ for which $[\sigma] = [\tau]$ in $KL(A, M(B)/B)$.*

(v) *If neither of τ and σ are weakly unital, then τ and σ are strongly approximately unitarily equivalent if and only if $[\sigma] = [\tau]$ in $KL(A, M(B)/B)$.*

Proof. Put $Q = M(B)/B$. Then Q is purely infinite and simple (see 2.6). In particular, Q has K_1-r-cancellation (for $r(n) = n$) and $\mathrm{cel}(M(B)/B) \le 2\pi + d$ (for any $d > 0$) (see [16]). To see (i), let $\xi = \pi_{*1}([\{u_n\}])$, where $\pi : l^\infty(Q) \to q_\infty(Q)$. By passing to a subsequence, without loss of generality, we may assume that $\xi \in H_1(K_0(A), K_1(q_\infty(Q)))$. Let $\Phi : A \to l^\infty(Q)$ be defined by $\Phi(a) = \{\sigma(a), \tau(a), \cdots, \}$ for all $a \in A$. Define $\bar\tau = \pi \circ \Phi$. It follows from 4.5 that there exists a full monomorphism $\bar\tau_1 : A \otimes C(S^1) \to q_\infty(Q)$ such that $[(\bar\tau_1)|_A] = [\Phi]$ in $KL(A, q_\infty(Q))$ and $[\bar\tau_1(1\otimes\imath)] = \xi$. It follows from 7.7 of [14] that Φ is approximately unitarily equivalent to $(\bar\tau_1)|_A$. As in the proof of 3.7 we may assume that $(\bar\tau_1)|_A$ is unital. There is a sequence of unitaries $w_n \in q_\infty(Q)$ such that

$$\lim_{n\to\infty} \mathrm{ad}\, w_n \circ \bar\tau_1(a \otimes 1) = \Phi(a) \ \text{ for all } \ a \in A.$$

Put $z = \bar\tau_1(1 \otimes \imath)$. Then

$$\lim_{n\to\infty} \|w_n^* z w_n \Phi(a) - \Phi(a) w_n^* z w_n\| = 0 \ \text{ for all } \ a \in A.$$

It is well known that there are unitaries $w_n(k), z(k) \in Q$ such that $\pi(\{w_n(k)\}) = w_n$ and $\pi(\{z(k)\}) = z$. Thus there is a subsequence $\{n(k)\}$ such that

$$\lim_{n\to\infty} \|w_{n(k)}(k)^* z(k) w_{n(k)}(k)\sigma(a) - \sigma(a) w_{n(k)}(k)^* z(k) w_{n(k)}\| = 0 \ \text{ for all } \ a \in A.$$

Let $v_k = w_{n(k)}(k)^* z(k) w_{n(k)}(k)$. Then $[v_k] = [z(k)]$ in $K_1(Q)$. By the definition of ξ, one checks that $[v_k] = [u_{n(k)}]$ for all sufficiently large k. Note that $v_k^* u_{n(k)} \in U_0(Q)$, since Q is purely infinite and simple. We have

$$\lim_{n\to\infty} \mathrm{ad}\, u_{n(k)} v_k^* \circ \tau(a) = \sigma(a) \ \text{ for all } \ a \in A.$$

This proves (i).

Note that (ii) follows from (i) and 2.5 immediately.

(iii) follows from the same proof of 4.4.

To see (iv), we note that, by (ii), we only need to show that the injection is actually surjective in this case. Suppose that $\{u_n\}$ be a sequence of unitaries such that

$$\sigma(a) = \lim_{n\to\infty} \mathrm{ad}\, u_n \circ \tau(a) \ \text{ for all } \ a \in A.$$

Let $\phi : K_1(M(B)/B) \to K_1(M(B)/B)/(\bar{G} + H_1(K_0(A), K_1(M(B)/B)))$ be the quotient map. By the assumption infinitely many $\phi([u_n])$ are the same. Suppose that $\{k(n)\}$ is a subsequence of $\mathbb{N}$ such that $\phi([u_{k(n)}]) = x$ for $n = 1, 2, ...$ for some $x \in K_1(M(B)/B)/(\bar{G} + H_1(K_0(A), K_1(M(B)/B)))$. Choose $u \in U(M(B)/B$ such that $\phi([u]) = x$. Then

$$\sigma(a) = \lim_{n\to\infty} \operatorname{ad} u_{k(n)} u^* \circ \operatorname{ad} u \circ \tau(a) \text{ for all } a \in A.$$

Since $[u_{k(n)}u^*] \in H_1(K_0(A), K_1(M(B)/B))$, by (i), σ is strongly unitarily equivalent to $\operatorname{ad} u \circ \tau$. This proves (iv).

For (v), by 2.5, we only need to show the "if" part. Since $M(B)/B$ is purely infinite simple C^*-algebra (see [13]), there is a unitary $v_1 \in U(M(B)/B)$ such that $v_1^*\tau_2(1)v_1 = \tau_1(1)$. Put $e = 1 - \tau_1(1)$. Again, since $M(B)/B$ is purely infinite and simple, there is a unitary $v_2' \in e(M(B)/B)e$ such that $[v_2'] = [v_1^*]$. Put $v_2 = v_2' + (1 - e)$. Then $v_2^* v_1^* \tau_2(1) v_1 v_2 = \tau_1(1)$. Note that $v_1 v_2 \in U_0(M(B)/B)$. Thus we may assume that $\tau_1(1) = \tau_2(1)$. Let $u_n \in M(B)/B$ be a sequence of unitaries such that

$$\lim_{n\to\infty} \operatorname{ad} u_n \circ \tau_1(a) = \tau_2(a) \text{ for all } a \in A.$$

Since $M(B)/B$ is purely infinite and simple, we obtain unitaries $w_n \in e(M(B)/B)e$ such that $[w_n] = [u_n]$. Put $z_n = w_n^* + (1 - e)$. Then $z_n u_n \in U_0(M(B)/B)$. One verifies that

$$\lim_{n\to\infty} \operatorname{ad} z_n \circ \tau_1(a) = \tau_2(a) \text{ for all } a \in A.$$

□

Remark 4.7. From the above theorem we also know that, for each $[\tau]$ in $KL(A, M(B)/B)$, there are at most $\prod_{n\in\mathbb{N}} K_1(M(B)/B)/H_1^{ap}(K_0(A), K_1(M(B)/B))$ many different strong approximate unitary equivalence classes of weakly unital essential extensions σ for which $[\sigma] = [\tau]$. However, given a sequence of elements $\{x_n\}$ in $K_1(M(B)/B)$, we do not know if there is a sequence of unitaries $\{u_n\}$ such that $[u_n] = x_n$ and $\operatorname{ad} u_n \circ \tau(a)$ converges for any $a \in A$. This prevents us from determining exactly how many different strong approximate unitary equivalence classes of weakly unital essential extensions σ with $[\sigma] = [\tau]$. In the case that $B = C \otimes \mathcal{K}$, one can make another estimate. There are at most $|K_0(B)/H_1(K_0(A), K_0(B))| \cdot |ext_{\mathbb{Z}}(K_*(A), K_*(B))|$ many strong approximate unitary equivalence classes of weakly unital essential extensions σ with $[\sigma] = [\tau]$. However, even if $[\sigma] \neq [\tau]$ in $KK^1(A, B)$ but $[\sigma] = [\tau]$ in $KL^1(A, B)$, they could still be strongly approximately unitarily equivalent. For example, this happens when $K_0(B) = H_1(K_0(A), K_0(B))$, since in this case, by 4.3, all approximately unitarily equivalent weakly unital full essential extensions are strongly approximately unitarily equivalent. Full extensions in different KK-classes could be strongly approximately unitarily equivalent even in the case that $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B)) \neq \{0\}$. For example, suppose that $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ is finite but $ext_{\mathbb{Z}}(K_1(A), K_0(M(B)/B))$ is infinite. In this case there are only finitely many strong approximate unitary equivalence classes of weakly unital essential extensions which give the same element in $KL^1(A, M(B)/B)$. Therefore there are infinitely many weakly unital essential extensions in different $KK(A, M(B)/B))$ (but in the same $KL(A, M(B)/B)$) that are strongly approximately unitarily equivalent.

On the other hand, the above results show that there are weakly unital full essential extensions which give the same element in $KK(A, M(B)/B)$ (or in $KK^1(A, B)$) may not strongly approximately unitarily equivalent as long as $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ is not trivial. For short,

$KK(A, M(B)/B)/KL(A, M(B)/B))$ can not be used to distinguish strong approximate unitary equivalence from approximate unitary equivalence. It is the group $H_1(K_0(A), K_1(M(B)/B)$ (or the approximate version of it) that detects the difference.

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